Inflation and nonequilibrium renormalization group

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Abstract. We study de spectrum of primordial fluctuations and the scale dependence of the inflaton spectral index due to self-interactions of the field. We compute the spectrum of fluctuations by applying nonequilibrium renormalization group techniques.

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In this paper we explore a mechanism to explain departures of the primordial fluctuations from the spectrum of a free inflaton field. This is important, because the spectrum of the inflaton field at the exit of the horizon is directly related with the level of inhomogeneity of the observed universe. The question we want to address is how the nonlinearities for an interacting field affect the predicted spectrum [1, 2, 3, 4]. To this end we will use a nonequilibrium renormalization group (RG).

If the inflaton field were a free field, its spectrum would be of Harrison-Zel'dovich type. At the horizon exit (HE) of the mode with wave number k it would be

$$\langle \Phi(k,t)\Phi(k,t)\rangle_{\mathrm{HE}} \propto \frac{1}{k^3}. \tag{1}$$

The spectral index $n(k)$ measures deviations from this law. It is defined by

$$\langle \Phi(k,t)\Phi(k,t)\rangle_{\mathrm{HE}} \propto \frac{1}{k^3}\, k^{n(k)-1}. \tag{2}$$

Hence, the Harrison-Zel'dovich spectrum (1) has $n(k) = 1$. Present experiments show that the spectral index is close to one and that presumably it runs with the scale,

$$n(k) = 1 + \Delta(k), \tag{3}$$

just at the edge of the experimental precision [5].

We will compute the spectrum of the inflaton field for a toy model of inflation. The goal is to show how a RG defined for nonequilibrium problems can be used to predict the spectrum of an interacting inflaton field. We will assume a spatially flat Robertson-Walker metric with constant expansion rate H,

$$ds^2 = dt^2 - a^2(t)\left(dx^2 + dy^2 + dz^2\right), \quad a(t) = a_0 e^{Ht}, \tag{4}$$

and an inflaton field described by a massless $\lambda\phi^4$ action

$$S[\Phi] = \int dt\ a(t)^3 \Bigg[\int d^dq\ \frac{1}{2} \left(\dot{\Phi}^2 - q^2 \frac{\Phi^2}{a(t)^2} + \xi R\ \Phi^2 \right) - \frac{\lambda}{4!} \int \prod_{i=1}^{4} d^dq_i\ \ \delta^3(\mathbf{q}_1 + \ldots + \mathbf{q}_4)\ \ \Phi_1\Phi_2\Phi_3\Phi_4 \Bigg]. \tag{5}$$

[$\Phi^2 \equiv \Phi(\mathbf{q}, t)\Phi(-\mathbf{q}, t)$, and $\Phi_i \equiv \Phi(\mathbf{q}_i, t)$.] Transforming to conformal coordinates

$$\eta = -(aH)^{-1}, \tag{6a}$$

$$a\Phi = \Phi_c, \tag{6b}$$

and ignoring all the mass terms, the original theory is mapped into a scalar field theory in flat space-time with time coordinate $-\infty < \eta < 0$,

$$S[\Phi_c] = \int d\eta \Bigg[\int d^dq\ \frac{1}{2} \left(\Phi_c'^2 - q^2\ \Phi_c^2 \right) - \frac{\lambda}{4!} \int \prod_{i=1}^{4} d^dq_i\ \ \delta^3(\mathbf{q}_1 + \ldots + \mathbf{q}_4)\ \ \Phi_{c1}\Phi_{c2}\Phi_{c3}\Phi_{c4} \Bigg], \tag{7}$$

where $\Phi_c' = \partial\Phi_c/\partial\eta$. We can show that if $\lambda = 0$ the expression (9) is regained. Note that when $\lambda = 0$, the expectation value for the product of two conformal fields Φ_c is given by the usual expression for a free field in flat space-time

$$\langle \Phi_c(k, \eta)\Phi_c(k, \eta) \rangle \propto \frac{1}{k}. \tag{8}$$

Using (6b), the expectation value for the original theory is

$$\langle \Phi(k, t)\Phi(k, t) \rangle \propto \frac{1}{a^2(t)k}. \tag{9}$$

To obtain the spectrum, this expression must be evaluated when the mode k exits the horizon. This happens when its physical wavelength $k^{-1}a(t)$ equals the horizon size H^{-1}, that is, when

$$a(t) = kH^{-1}. \tag{10}$$

Thus, replacing this value of a in (9), Eq. (1) is recovered. We will use RG techniques to compute the spectrum when $\lambda \neq 0$.

The basic idea of RG for systems in equilibrium (where time does not enter in the description) is the coarse graining of the original system, i.e. the change in the resolution with which the system is observed [6]. Given a system with a range of scales which goes up to wave number Λ, if we are only interested in scales up to wave number $k < \Lambda$, we can separate the original system in two sectors: a lower wave number sector, with $k' < k$, the relevant system, and a higher wave number sector with $k < k' < \Lambda$, the environment. Once this division is done, the environment modes are eliminated from the description. In equilibrium, this is achieved by computing the coarse grained 'in-out' effective action for the lower sector, complemented with a rescaling of the fields and momenta that restores the cutoff and the coefficient of the q^2 term in the action to their initial values. The elimination of the modes between Λ and k proceeds by infinitesimal steps. In this way, the calculation involves only tree and one loop diagrams, and the resulting equations form a set of differential equations for the parameters that define the effective action [7].

Essentially, the same scheme can be used for nonequilibrium systems. The main difference with the usual approach is that the time variable must enter in the description. We want to compute true expectation values at given times, not transition amplitudes between 'in' and 'out' asymptotic states, far away in the future and in the past. We want to follow the real and causal evolution of expectation values, for which the usual 'in-out' representation is not appropriate. A suitable description of nonequilibrium systems is given within the 'closed time path' (CTP) formalism [8, 9, 10, 11, 12]. The number of fields is doubled, path integrals are over two histories φ^+ and φ^-, that coincide at the time of observation T, which is a new dimensional parameter of the theory.

The possibility of couplings between the two histories enlarge the parameter space. Notably, it includes noise and dissipation. Written in terms of $\phi = \varphi^+ - \varphi^-$ and $\varphi = \varphi^+ + \varphi^-$, the free action for a scalar field in the CTP formalism is

$$S_0[\phi,\varphi] = \int_0^T dt \int d^dq \left[\frac{1}{2}\dot{\phi}\,\dot{\varphi} - \frac{1}{2}\left(q^2 + m^2\right)\phi\varphi - \kappa\,\phi\dot{\varphi} + \frac{i}{2}\nu\,\phi\phi\right]. \tag{11}$$

[$\phi\varphi \equiv \phi(\mathbf{q},t)\varphi(-\mathbf{q},t)$, etc.] Here κ is associated to dissipation and ν to noise. For a given order n in the fields, there are n possible interaction terms. Thus, for example, the quartic interactions that can appear in the CTP action are

$$\begin{aligned}\int_0^T \prod_{i=1}^4 dt_i \int \prod_{i=1}^4 d^dq_i \Big[& v_{41}(q,t)\ \phi_1\ \varphi_2\varphi_3\varphi_4 + i\, v_{42}(q,t)\ \phi_1\phi_2\ \varphi_3\varphi_4 \\ & + v_{43}(q,t)\ \phi_1\phi_2\phi_3\ \varphi_4 + i\, v_{44}(q,t)\ \phi_1\phi_2\phi_3\phi_4\ \Big].\end{aligned} \tag{12}$$

[$\phi_i = \phi(q_i,t_i)$, etc.] In principle, all the allowed couplings must be taken into account to compute the RG equations.

Even when the initial action at scale Λ is the usual, local, massless, $\lambda\varphi^4$ action, which in the CTP representation reads

$$S_\Lambda = \int_0^T dt \left[\int d^dq\, \frac{1}{2}\left(\dot{\phi}\,\dot{\varphi} - q^2\phi\,\varphi\right) - \frac{\lambda}{48}\int \prod_{i=1}^4 d^dq_i\left(\phi_1\ \varphi_2\varphi_3\varphi_4 + \phi_1\phi_2\phi_3\ \varphi_4\right)\right], \tag{13}$$

as short wave number modes are eliminated out, the RG flow generates all the possible terms, with an arbitrary number of fields and nonlocal dependencies (but with certain constrains imposed by the CTP formalism). However, after a brief excursion, most of these terms will go to zero along with λ, except for a few terms in the free action. The effective action at scale k will be given approximately by (11), but its parameters will depend on both, the scale k and the time T

$$\begin{aligned}S_k \sim \int_0^T dt \int d^dq \Bigg\{ & \frac{1}{2}\dot{\phi}\,\dot{\varphi} - \frac{1}{2}\left[q^2 + \left(\frac{\Lambda}{k}\right)^2 m^2(k,T)\right]\phi\,\varphi \\ & - \left(\frac{\Lambda}{k}\right)\kappa(k,T)\,\phi\,\dot{\varphi} + \frac{i}{2}\left(\frac{\Lambda}{k}\right)^2 \nu(k,T)\phi^2\Bigg\}.\end{aligned} \tag{14}$$

Here we have extracted from m^2, κ, and ν the factors merely induced by the rescaling. The influence of the environment on the spectrum of the long wave modes will manifest through these terms [13, 14, 15, 16].

The flow of the RG drives the initial interacting theory (13) towards the free theory (14), and allow us to find a relation between expectation values associated with each theory. The relation is

$$G\Big(k, t, \mu(\Lambda, T)\Big) = (\Lambda/k)^{\alpha(k,T)}\ G\left(\Lambda,\ (\Lambda/k)^{\beta(k,T)}\ t, \mu(k,T)\right). \tag{15}$$

On the left hand side, G is the two field expectation value computed for a mode k at time t, and $\mu(\Lambda, T)$ stands for the set of parameters which define the action at scale Λ. In our case the only parameter is the coupling constant λ. On the right hand side, G is the expectation value of the theory defined by the set of parameters $\mu(k, T)$, reached after modes between k and Λ have been eliminated. The relevant parameters in $\mu(k, T)$ are $m^2(k, T)$, $\kappa(k, T)$, and $\nu(k, T)$. Finally, the exponents α and β depend on the trajectory followed by the action when it goes from scale Λ to k.

Now we connect to the original problem for the power spectrum of an interacting inflaton field. We must feed the RG group equations with an initial condition at scale Λ and then use the relation (15) to obtain the expectation value for the mode k as it exits the horizon. The initial condition, in terms of the conformal field, is given by the CTP action at scale Λ, Eq.(13), where t has to be substituted by the conformal time η. According to Eq.(10), the mode k exits the horizon when

$$\eta = -k^{-1}. \tag{16}$$

If inflation starts at η^*, the time that the mode k spends inside the horizon is given by

$$\tau_k = -k^{-1} - \eta^*. \tag{17}$$

For the interacting theory Eq.(9), reads

$$\langle \Phi(k,t)\Phi(k,t)\rangle_{\mathrm{HE}} = k^{-2}\ G\Big(k, \tau_k, \lambda\Big). \tag{18}$$

From Eq.(15), identifying t and T with τ_k, we get

$$\begin{aligned}\langle \Phi(k,t)\Phi(k,t)\rangle_{\mathrm{HE}} &= k^{-2}\ (\Lambda/k)^{\alpha(k,\tau_k)} \\ &\times G\left(\Lambda,\ (\Lambda/k)^{\beta(k,\tau_k)}\ \tau_k, \Big\{m^2(k,\tau_k), \kappa(k,\tau_k), \nu(k,\tau_k)\Big\}\right). \end{aligned} \tag{19}$$

Here, the relevant elements of $\mu(k, \tau_k)$ have been shown explicitly. The right hand side of Eq.(19) can be calculated using the G corresponding to the action (14)

$$G\left(k, t, \Big\{m^2, \kappa, \nu\Big\}\right) = \left(\frac{2}{k} - \frac{\nu}{\kappa\omega_0^2}\right)\left[\frac{\omega_0^2}{\omega^2} - \frac{\kappa^2}{\omega^2}\cos(2\omega t) + \frac{\kappa}{\omega}\sin(2\omega t)\right] e^{-2\kappa t} + \frac{\nu}{\kappa\omega_0^2}, \tag{20}$$

where $\omega_0^2 = m^2 + k^2$ and $\omega^2 = \omega_0^2 - \kappa^2$ [17].

The expressions for m^2, κ, and ν, and for the exponents α and β in Eq. (19), as functions of k and τ_k are given in [17]. In Fig. 1 we show $n(k) - 1$, defined in (2), as function of k for a particular choice of λ and η^*. (We have chosen $-\eta^* = 200$, large enough so the ratio $k_{\mathrm{max}}/k_{\mathrm{min}}$ can be about 100, the expected ratio between the maximum and minimum length scales of the inhomogeneities.) The main effects are introduced by the mass term.

We have presented a toy model for computing the spectrum of the fluctuations of the inflaton field. At the present stage we do not pretend to derive quantitative

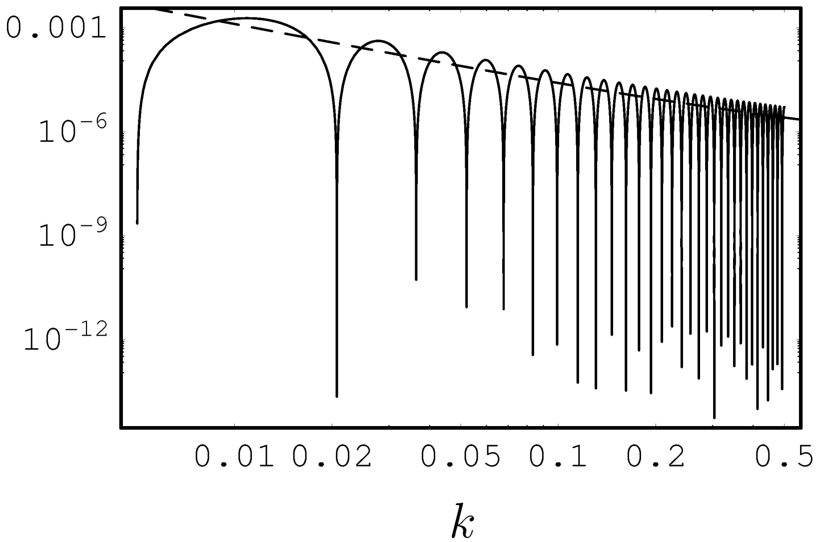

Figure 1. $n(k)-1$ as function of k (solid curve), with $\eta^* = -200$, and $\lambda = 10^{-3}$. The dashed curve shows $n(k)-1$ once the rapidly oscillatory terms have been removed. The main departures from zero come from the mass term induced by the coarse graining.

conclusions from this model. Our main concern was to show that the same arguments usually given in the context of the RG in equilibrium, can be extended to nonequilibrium problems using the CTP formalism, where noise and dissipation show up naturally.

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References

[1] Calzetta E and Hu B L 1995 *Phys. Rev.* D **52** 6770
[2] Calzetta E and Gonorazky S 1997 *Phys. Rev.* D **55** 1812
[3] Boyanovsky D, de Vega H J and Sanchez N G 2006 Clarifying Slow Roll Inflation and the Quantum Corrections to the Observable Power Spectra *Preprint* astro-ph/0601132
[4] Wu C.-H, Ng K.-W, Lee W, Lee D.-S and Charng Y.-Y 2006 Quantum noise and a low cosmic microwave background quadrupole *Preprint* astro-ph/0604292
[5] Ballesteros G, Casas J A and Espinosa J R 2006 *J. Cosmol. Astropart. Phys.* JCAP03(2006)001
[6] Wilson K and Kogut J 1974 *Phys. Rep.* **12** 75
[7] Wegner F and Houghton A 1973 *Phys. Rev.* A **8** 401
[8] Schwinger J 1961 *J. Math. Phys.* **2** 407
[9] Keldysh L V 1964 *Zh. Eksp. Teor. Fiz.* **47** 1515 [Engl. trans: Keldysh L V 1965 *Sov. Phys.* JEPT **20** 1018]
[10] Calzetta E and Hu B H 1987 *Phys. Rev.* D **35** 495
[11] Calzetta E and Hu B H 1988 *Phys. Rev.* D **37** 2878

[12] Dalvit D A R and Mazzitelli F D 1996 *Phys. Rev.* D **54** 6338

[13] Starobinsky A 1986 *Current Topics in Field Theory, Quantum Gravity and Strings* (*Lecture Notes in Physics* vol 246) eds de Vega H J and Sanchez N (Springer: Berlin)

[14] Lombardo F C and López Nacir D 2005 *Phys. Rev.* D **72** 063506

[15] Matarrese S, Musso M A and Riotto A 2004 *J. Cosmol. Astropart. Phys.* JCAP05(2004)008

[16] Liguori M, Matarrese S, Musso M A and Riotto A 2004 *J. Cosmol. Astropart. Phys.* JCAP08(2004)011

[17] Zanella J and Calzetta E 2006 Renormalization group study of damping in nonequilibrium field theory *Preprint* hep-th/0611222